We find, however, that the expressions given in /2/ for the characteristic quantities

$$V_{ij}' = \frac{3}{4}\frac{a}{r}(n_i n_j + \delta_{ij}) - \frac{3}{4}\left(\frac{a}{r}\right)^3\left(n_i n_j - \frac{1}{3}\delta_{ij}\right) \qquad (A.4)$$

$$P_i' = \frac{3}{2}\frac{a}{r^2}n_i, \quad n_i = \frac{x_i}{r}$$

do not satisfy Eqs.(A.2). Therefore the expressions given for the coefficients of resistance of a spherical particle are incorrect.

The error in /2/ is caused by the fact that instead of Eq.(A.2) the author used

$$\nabla_l P_s' = \nabla_j \nabla_j V_{ls}' \qquad (A.5)$$

which were obtained as follows. In order for the pair (V_{is}', P_s') to be a solution of (A.2), it is sufficient for that pair to be a solution of the equation $\nabla_j(-\delta_{lm}P_s' + \nabla_m V_{ls}' + \nabla_l V_{ms}') = 0$, or in expanded form, to

$$\begin{aligned}
&j = m = 1, \quad -\delta_{l1}\nabla_1 P_s' + \nabla_1\nabla_1 V_{ls}' + \nabla_l\nabla_1 V_{1s}' = 0 \\
&j = m = 2, \quad -\delta_{l2}\nabla_2 P_s' + \nabla_2\nabla_2 V_{ls}' + \nabla_l\nabla_2 V_{2s}' = 0 \\
&j = m = 3, \quad -\delta_{l3}\nabla_3 P_s' + \nabla_3\nabla_3 V_{ls}' + \nabla_l\nabla_3 V_{3s}' = 0 \\
&j = 1, \quad m = 2, \quad -\delta_{l2}\nabla_1 P_s' + \nabla_1\nabla_2 V_{ls}' + \nabla_l\nabla_1 V_{2s}' = 0 \\
&j = 1, \quad m = 3, \quad -\delta_{l3}\nabla_1 P_s' + \nabla_1\nabla_3 V_{ls}' + \nabla_l\nabla_1 V_{3s}' = 0 \\
&j = 2, \quad m = 2, \quad -\delta_{l1}\nabla_2 P_s' + \nabla_1\nabla_2 V_{ls}' + \nabla_l\nabla_2 V_{1s}' = 0 \text{ etc.}
\end{aligned} \qquad (A.6)$$

In /2/ the first three equations of (A.6) were combined to obtain, naturally, (A.5), and the remaining equations of (A.6) were neglected. The present discussion shows, however, that the system (A.3), (A.5) is not equivalent to the system (A.2), (A.3).

We note that in the special case of isotropic viscosity

$$\eta_{ijlm} = \eta(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})$$

the relations (A.1) become $v_i = V_{is}'u_s$, $p = P_s'u_s\eta$, equations (A.2) reduce to Eqs.(A.5) and the problem, as well as the method of solving it in /2/, become identical with the results in /4/.

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ON THE STABILITY OF A VAPOUR-LIQUID MEDIUM CONTAINING BUBBLES*

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A problem of the stability of a vapour-liquid medium containing bubbles is investigated. It is shown that since the surface tension and phase transitions act simultaneously, a range of values of the parameters of the vapour-liquid and vapour-gas-liquid media containing bubbles exists, for which the equilibrium state is unstable. The effect of various parameters of the two-phase medium, such as the volume content of the bubbles, the mass content of the gas and the degree of dispersion of the medium, on the increment characterizing the rate of development of the instability, is analysed.

1. Fundamental equations. Let us consider the propagation of small perturbations through a polydisperse mixture of liquid and bubbles of $m-1$ kinds, under the usual assumptions made for two-phase media. Moreover, we shall assume that the gaseous phase consists of the vapour from the liquid phase, and some "inert" gas which takes no part in the process

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of mass transfer between the phases. The phase velocities are the same. Then the system of equations of the mass, the number of bubbles and the momentum for one-dimensional motion, has the following form in the linear approximation:

$$\frac{\partial\rho_1}{\partial t}+\rho_{10}\frac{\partial v}{\partial x}=-\sum_{i=2}^{m}I_i,\quad \frac{\partial\rho_i}{\partial t}+\rho_{i0}\frac{\partial v}{\partial x}=I_i \tag{1.1}$$
$$I_i=4\pi a_{i0}^2 n_{i0} j_i,\quad \frac{\partial n_i}{\partial t}+n_{i0}\frac{\partial v}{\partial x}=0,\quad \rho_0\frac{\partial v}{\partial t}=-\frac{\partial p_1}{\partial x}$$
$$\rho=\sum_{i=1}^{m}\rho_i,\quad \rho_i=\rho_i^{\circ}\alpha_i,\quad \alpha_i=\frac{4}{3}\pi a_i^3 n_i$$

The indices $i=1$ and $i=2,\ldots,m$ refer to the parameters of the liquid and gas respectively in bubbles of the i-th kind, $\rho_i, \rho_i^0, v, p_i, n_i, a$ are the perturbations in the density, velocity, pressure, number of bubbles per unit volume of the mixture and the radius of the bubbles, and I_i, j_i are the mass transfer intensities between the phases per unit volume of the mixture and unit area of the interphase boundary. The parameters corresponding to the unperturbed state, have an additional zero subscript.

In order to take into account the heat and mass transfer between the phases, we will write the equation of heat conduction inside and outside the bubbles, and the diffusion equation inside the bubble, as

$$\rho_{10}^{\circ}c_1\frac{\partial T_{1i}'}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_1 r^2\frac{\partial T_{1i}'}{\partial r}\right)\quad (r>a_{i0}) \tag{1.2}$$
$$\rho_{i0}c_{ip}\frac{\partial T_i}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_i r^2\frac{\partial T_i'}{\partial r}\right)+\frac{\partial p_i}{\partial t}-\rho_{i0}(B_v-B_g)T_0\frac{\partial g_i'}{\partial t}$$
$$\frac{\partial g_i'}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\varkappa_i r^2\frac{\partial g_i'}{\partial r}\right),\quad c_{ip}=c_{vp}g_{i0}+c_{gp}(1-g_{i0}),$$
$$i=2,\ldots,m\quad (r<a_{i0})$$

Here r is the microcoordinate (the distance from the bubble centre), T', g' is the temperature and mass vapour content distribution, c_1, c_p, c_v is the specific heat capacity of the liquid and of the mixture in bubbles at constant pressure and volume respectively, B is the gas constant, λ_i is the thermal conductivity of the phases, and $\varkappa_i$ is the diffusion coefficient; the primes denote the microparameters (i.e. the parameters depending on r).

We write the equations of state in the form

$$\rho_1^{\circ}=\rho_{10}^{\circ},\quad p_i=p_{vi}'+p_{gi}'=(\rho_v'B_v+\rho_g'B_g)T_i'=\rho_i^{\circ\prime}B_iT_i',\quad i=2,\ldots,m \tag{1.3}$$

where the subscripts v and g refer to the vapour and the gas within the bubbles.

The equation of oscillatory motion, ignoring the compression of the bubbles, will be written in the form

$$a_{i0}\partial w_{1i}/\partial t+4\nu_1 w_{1i}/a_{i0}=(p_i-p_1+2\sigma/a_i)/\rho_{i0}^{\circ} \tag{1.4}$$

where w_{1i}, w_i are the velocities of the oscillatory motion of the liquid and gas at the interphase surface, σ is the surface tension and ν_1 is the kinematic viscosity of the liquid.

We will specify the following conditions at the surface of separation of the phases $(r=a_{i0})$:

$$T_{1i}'=T_i'=T_{ia},\quad \lambda_1\frac{\partial T_{1i}'}{\partial r}-\lambda_i\frac{\partial T_i'}{\partial r}=j_i l \tag{1.5}$$
$$\rho_{i0}^{\circ}\left[(1-g_{i0})\left(\frac{\partial a_i}{\partial t}-w_i\right)-\varkappa_i\frac{\partial g_i'}{\partial r}\right]=0$$
$$\rho_{i0}^{\circ}\left[g_{i0}\left(\frac{\partial a_i}{\partial t}-w_i\right)+\varkappa\frac{\partial g_i'}{\partial r}\right]=\rho_{10}^{\circ}\left(\frac{\partial a_i}{\partial t}-w_{1i}\right)=j_i,\quad i=2,\ldots,m$$

where l is the latent heat of vapourisation. Moreover we have

$$\partial g_i'/\partial r=0,\ \partial T_i'/\partial r=0\ (r=0),\ T_{1i}'=T_0\ (r=\infty) \tag{1.6}$$

We will write the Clausius-Clapeyron equation for the values of the parameters at the boundary of separation of the phases, for states distant from the critical states as

$$dp_{vi}/\partial T_{ia}=\rho_{vj}l/T_{ia}$$

In the course of solving the problems it is convenient to use the relation for p_i obtained from the equation of heat conduction inside the bubbles, under the assumption that the condition of uniform pressure /1/ holds

$$\frac{\partial p_i}{\partial t}=-\frac{3}{a_{i0}}\left[\gamma_i p_{i0}w_i-\rho_{i0}^{\circ}(B_v-B_g)T_0\varkappa_i\left(\frac{\partial g_i'}{\partial r}\right)_{a_{i0}}-\right. \tag{1.7}$$
$$\left.(\gamma_i-1)\lambda_i\left(\frac{\partial T_i'}{\partial r}\right)_{a_{i0}}\right],\quad \gamma_i=\frac{c_{ip}}{c_{iv}},\quad i=2,\ldots,m$$

We will seek the solution for the system in question in the form $p, v, a, w \sim \exp(iKx + \omega t)$, $T' = T(r)\exp(iKx + \omega t)$, $g' = g(r)\exp(iKx + \omega t)$ where K is the wave number and ω is the complex frequency. Assuming that the effect of radial inertia and viscosity can be neglected as compared with the effect of the lack of thermal equilibrium, i.e. assuming that the extent of the instability is determined by the heat and mass transfer processes, we obtain the following dispersion equation:

$$\frac{K^2}{\omega^2} + 3\rho_0 \sum_{i=2}^{m} \frac{\alpha_{i0}}{\psi_i} = 0, \quad \psi_i = \frac{3\gamma_i p_{i0}}{Q_i} - \frac{2\sigma}{a_{i0}} \tag{1.8}$$

$$Q_i = 1 + \left[\frac{\gamma_i H_{vi} \Pi(z_i)}{1 - g_{i0}} + \frac{(\gamma_i - 1) H_{gi} \Pi(x_i)}{g_{i0}}\right] \left[\frac{H_{gi}}{g_{i0}} + \frac{\gamma_i \Pi(z_i) y_i^2}{(1 - g_{i0})\beta_i (1 + y_i)}\right]^{-1}$$

$$\Pi(x) = 3(x \operatorname{cth} x - 1)x^{-2}, \quad x_i = (\omega a_{i0}^2/\varkappa_i^{(T)})^{1/2}$$

$$y_i = (\omega a_{i0}^2/\varkappa_1^{(T)})^{1/2}, \quad z_i = (\omega a_{i0}^2/\varkappa_i)^{1/2}, \quad \varkappa_1^{(T)} = \lambda_1/(\rho_{10}{}^{\circ} c_1)$$

$$\varkappa_i^{(T)} = \lambda_i/(\rho_{i0}{}^{\circ} c_{ip}), \quad H_{vi} = B_v/B_{i0}, \quad H_{gi} = B_g/B_{i0}, \quad p_{i0} = p_{10} + 2\sigma/a_{i0}$$

$$\beta_i = 3(\gamma_i - 1) H_{vi} \frac{\rho_{10} c_1}{\rho_{i0}{}^{\circ} c_{ip}} \left(\frac{c_{ip} T_0}{l}\right)^2, \quad g_{i0} = \left[1 + \frac{B_v}{B_g}\left(\frac{p_{i0}}{p_{v0}} - 1\right)\right]^{-1}$$

2. A vapour-liquid medium ($g_{i0} = 1$). In the present case the equilibrium mixture will always be monodisperse and relation (1.8) will take the form

$$f(\omega) = \left[1 + \frac{\beta(1 + y)}{y^2}\right]^{-1} + \left(\frac{\omega}{Kd}\right)^2 - \Sigma = 0 \tag{2.1}$$

$$\left(d^2 = \frac{\gamma p_{20}}{\rho_{10}{}^{\circ} \alpha_{10} \alpha_{20}}, \quad \Sigma = \frac{2\sigma}{3\gamma p_{20}}\right)$$

The function $f(\omega)$ satisfies on the positive semi-axes the conditions $f'(\omega) > 0$, $f(0) = -\Sigma < 0$; $f(\omega) \to \infty$ as $\omega \to +\infty$. Therefore Eq.(2.1) has a unique positive root. Using the argument principle /2/ we can show that there are no other solutions in the complex right half-plane.

We will show that in the left half-plane Eq.(2.1) has complex conjugate roots corresponding to two running decaying waves moving in opposite directions. To do this, we take in the left upper quadrant the contour shown in Fig.1a. Introducing the parameter $\omega a_0^2/\varkappa_1 = -x^2$ we have, on the segment NW,

$$\operatorname{Im}(f) = \frac{\beta x^3}{(\beta - x)^2 + \beta^2 x^2}$$

Therefore on the segment NW $\operatorname{Im}(f) > 0$ and the contour changes its shape to the one shown in Fig.1b. Hence the function $f(\omega)$ has a single root within the region bounded by this contour. By virtue of the properties of $f(\omega)$, the complex conjugate point will also be a root of (2.1).

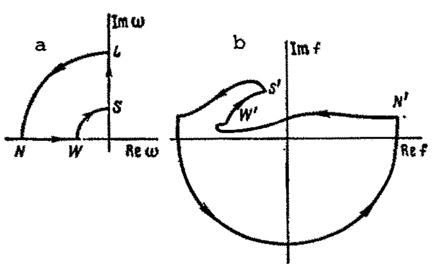

Fig.1

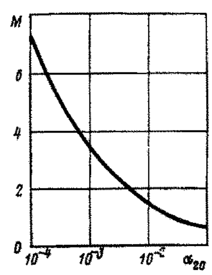

Fig.2

An analysis of (2.1) shows that when the wave number K varies from zero to infinity, the increment ω increases monotonically from zero to some maximum value ω_∞, and

$$\omega_\infty/\omega^{(T)} = 4\left[(1 + 4(\Sigma^{-1} - 1)/\beta)^{1/2} - 1\right], \quad \omega^{(T)} = \varkappa_1^{(T)}/a_0^2 \tag{2.2}$$

Note that the following physical meaning can be assigned to the parameter y: it represents the ratio of the bubble radius to the liquid layer thickness near the interphase surface where the temperature fluctuations mostly occur. Let $y \ll 1$. Then, solving Eq.(2.1) for the increment, we have

$$\omega = \frac{K^2 d^2}{2\omega_*}\left[\left(1 + \frac{4\omega_*^2\Sigma}{K^2 d^2}\right)^{1/2} - 1\right] \quad (\omega_* = \omega^{(T)}\beta) \tag{2.3}$$

and this yields

$$\omega \simeq K d \sqrt{\Sigma}, \quad K^2 \ll K_\infty^2 = 4\omega_*^2\Sigma/d^2$$
$$\omega \simeq \omega_\infty (1 - \omega_*^2\Sigma/(K^2 d^2)), \quad K^2 \gg K_\infty^2 \quad (\omega_\infty = \omega_*\Sigma)$$

The assumptions used in deriving (2.3) imply, that $(\beta\Sigma)^{1/2} \ll 1$. Then we obtain the following expression for the bubble radius:

$$a_0^{1/2} \gg a_*^{1/2}, \quad a_* = 2\sigma\beta/(3\gamma p_{20}) \tag{2.4}$$

When $p_{10} = 10^5$ and 10^6 Pa, for a water vapour-water mixture, for example, we have $a_* = 10^{-4}$ and $1.5\cdot 10^{-6}$ m. In the analysis of the stability the quantity ω_∞ appears to be the most important. For the pressure values given above and for $a_0 = 10^{-4}$ m we have, according to (2.2), $\omega_\infty = 50$ and $0.3\,\mathrm{sec}^{-1}$, and for $a_0 = 10^{-3}$ m we have $\omega_\infty = 0.2\cdot 10^{-1}$ and $0.3\cdot 10^{-3}$. Thus the water vapour-water mixtures are strongly unstable when $a_0 \lesssim 10^{-4}$ m.

The boundary condition (1.6) for T_1' expressing the constancy of the temperature away from the bubble (the condition that the cell is isothermal /1/) is adopted in some cases when the temperature fluctuations in the fluid initiated by the radial motions of the bubbles are smaller than the mean distance separating the bubbles $(y(A-1) \gg 1,\ A = \alpha_{20}^{-1/3})$. When $y(A-1) \ll 1$, the above boundary condition must be replaced by the condition of adiabaticity of the cell

$$\partial T_1'/\partial r = 0 \quad (r = a_0\alpha_{20}^{-1/3})$$

where $a_0\alpha_{20}^{-1/3}$ is the radius of the spherical cell, and in place of (2.1) we obtain

$$3\alpha_{20}\{\beta\alpha_{10}[1 - M^2y^2(A-1)^2]\}^{-1} + (\omega/(Kd))^2 - \Sigma = 0$$
$$M = (A-1)(5A^3 + 6A^2 + 3A + 1)/[15(A^3 - 1)] \tag{2.5}$$

The above equation has a positive solution if $\alpha_{20}/\alpha_{10} < \beta\Sigma/3$, otherwise it has no solutions in the right half-plane and the mixture is therefore stable. The dependence of the increment on the wave number is the same as before. We see from the relation $M(\alpha_{20})$ shown in Fig.2 that within the framework of the assumptions used in deriving (2.5), we can assume that $My^2(A-1)^2 \ll 1$ when $\alpha_{20} \gtrsim 10^{-3}$. Then, solving Eq.(2.5) for the increment, we obtain

$$\omega = \frac{3}{2}\frac{(Kd)^2}{\omega^{(T)}}\frac{M(A-1)^2\alpha_{20}}{\beta\alpha_{10}}\left[\left(1 + \frac{4}{3}\frac{\omega^{(T)}\omega_\infty}{(Kd)^2}\frac{\beta\alpha_{20}}{M(A-1)^2\alpha_{20}}\right)^{1/2} - 1\right] \tag{2.6}$$
$$\frac{\omega_\infty}{\omega^{(T)}} = \left(\frac{\beta\Sigma\alpha_{10}}{3\alpha_{20}} - 1\right)[M(A-1)^2]^{-1}$$

The above solution satisfies the conditions used in deriving it, provided that $\beta\Sigma\alpha_{10}/(3\alpha_{20}) - 1 \ll 1$. When $p_{10} = 10^5$ and 10^6 Pa and $a_0 = 10^{-3}$ m, we have $\beta\Sigma = 0.105;\ 0.0135$, and we can therefore use the solution (2.6) for the volume content of the bubbles close to the values $\alpha_{20} = 0.35\cdot 10^{-1};\ 0.45\cdot 10^{-2}$.

Let us consider (2.1) for $y \gg 1$. Then, remembering that we usually have $\beta \gg 1$, we obtain

$$(1 + \beta/y)^{-1} < (\omega/(Kd))^2 - \Sigma = 0$$

This yields the following expression for the maximum value of the increment:

$$\omega_\infty/\omega^{(T)} = [\beta\Sigma/(1 - \Sigma)]^2$$

The solution holds for sufficiently finely dispersed media such that $a_0 < a_*$, where a_* is the characteristic radius given by (2.4).

To explain the mechanism of the instability under discussion, we will consider a vapour-liquid mixture containing bubbles, in the isothermal equilibrium approximation ($T_1' = T_2' = T_0 =$ const), i.e. we will consider a hypothetical mixture for which the thermal conductivities are infinite. Then the pressure within the liquid and the bubbles will be connected by the relation $p_2 = p_1 + 2\sigma/a$. Since $p_2 = p_S(T_2)$ ($p_S(T_2)$ is the saturation pressure at the temperature T_2), it follows that the vapour pressure within the bubbles is constant in this approximation. Consequently the pressure perturbations δp_1 and the perturbations in the bubble radius are connected by the relation

$$\delta p \simeq \delta p_1 = 2\sigma a_0^{-2}\delta a \tag{2.7}$$

Replacing δa by the mean density perturbation of the medium and neglecting the compressibility of the liquid, we obtain $\delta p = -d^2\Sigma\delta\rho$ in place of (2.7). Thus when the "perfect" medium in question is compressed $(\delta\rho > 0)$, it responds by a drop in pressure and is therefore unstable.

On the other hand, by replacing the isothermal condition by the cell adiabaticity condition, we obtain a narrowing of the domain of instability. For such an equilibrium mixture the relation connecting the increment with the wave number K is linear: $\omega = d\sqrt{\Sigma}K$. Taking account of the lack of equilibrium in the heat and mass transfer processes, radial inertia and other effects, only perturbs the linear form of this relation without affecting the domain of instability.

3. A vapour-gas-liquid mixture. Using the assumptions and simplifications noted above, above, we can reduce the dispersion equation in the monodisperse approximation to the form

$$\left[1+\left(\frac{\gamma H_v \Pi(z)}{1-g_0}+\frac{(\gamma-1)H_y\Pi(x)}{g_0}\right)\left(\frac{H_g}{g_0}+\frac{\gamma\Pi(z)y^2}{(1-g_0)\beta(1+y)}\right)^{-1}\right]^{-1}+\left(\frac{\omega}{Kd}\right)^2-\Sigma=0 \qquad (3.1)$$

We can show, as before, that the above equation has a positive root, provided that

$$g_0>g_*,\quad g_*=\varphi/(1+\varphi),\quad \varphi=(2+3p_{10}a_0/(2\sigma))B_gB_v^{-1} \qquad (3.2)$$

If on the other hand the condition is rewritten in terms of the partial vapour pressure within the bubbles, we have

$$p_{v0}=p_S(T_0)>p_{10}+4\sigma/(3a_0) \qquad (3.3)$$

Let us consider the asymptotic form of Eq.(3.1) when $y\ll 1$. Then since we usually have $\varkappa_1^{(T)}\varkappa^{-1}\ll 1$, $\varkappa_2^{(T)}\varkappa^{-1}\sim 1$, the conditions $x, z\ll 1$ will also hold. Using the following expansion for the transcendental expression:

$$\Pi(x)=1+O(x^2/15)$$

and neglecting y^2/β as compared with γ, we can write Eq.(3.1) in the form

$$y^2g_0/\beta+(\omega/Kd))^2+(1-g_0)H_g\gamma^{-1}-\Sigma=0 \qquad (3.4)$$

from which we have

$$\omega_\infty=\omega_*(p_{v0}-p_{10}-4\sigma/(3a_0))/(\gamma p_{20}g_0) \qquad (3.5)$$

The above solution satisfies the conditions under which it was obtained, for sufficiently large bubbles satisfying condition (2.7).

The solution satisfying the conditions $y\gg 1$ and at the same time $x, z\leqslant 1$, is also of interest. In this case we have

$$\omega_\infty=\omega^{(T)}[\beta(p_{v0}-p_{10}-4\sigma/(3a_0))/(\gamma p_{20}g_0(1-\Sigma)) \qquad (3.6)$$

When water is mixed with water vapour-air bubbles, $p_{10}=10^4, 10^5, 10^6$ Pa and $a_0=10^{-4}$ m, we have $g_*=0.94;\ 0.994;\ 0.9994$ for the critical concentration of the vapour-air mixture. Thus although, as shown in Sect.2, the bubble mixtures are strongly unstable when $a_0\lesssim 10^{-4}$ m, adding a negligible amount of gas to the vapour bubbles stabilizes them.

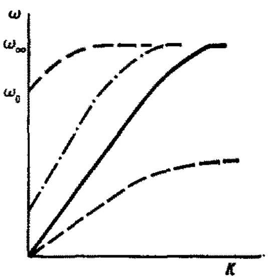

Fig.3

As before, replacing the isothermal condition by the cell adiabaticity condition we can sharpen the condition of stability, which has the form

$$(1-g_0)H_g\gamma^{-1}+3g_0\alpha_{20}/(\beta\alpha_{10})>\Sigma$$

We see that the correction on account of the condition of adiabaticity becomes substantial if $\beta\Sigma\alpha_{10}/(3\alpha_{20})\sim 1$.

Analysing the solutions (3.5) and (3.6) obtained for $x, z\leqslant 1$, we find that they satisfy this condition over a fairly wide range of variations of the parameters of the vapour-gas-liquid mixtures containing bubbles. We note that the diffusion coefficient and thermal conductivity of the vapour-gas mixture do not appear in these solutions, and therefore the pace of instability (the value of ω) is limited by the thermal resistance of the liquid.

Let us consider a mixture containing bubbles of two different sizes, assuming that the larger bubbles (radii a_{20}) contain vapour only. Then, when $y_2, y_3\ll 1$, we can write (1.8) in the form

$$(K/\omega)^2+[d_2^2(y_2^2/\beta_2-\Sigma_2)]^{-1}+[d_3^2(y_3^2g_0/\beta_3-\Sigma_3+(1-g_0)H_g/\gamma_3)]^{-1}=0 \qquad (3.7)$$
$$d_i^2=\gamma_ip_{i0}/(\rho_{10}\alpha_{10}\alpha_{i0}),\quad \Sigma_i=2\sigma/(3\gamma_ip_{i0}),\quad i=2,3$$

Analysing (3.7) we find that two types of dependence of the increment on the wave number are possible. If the vapour mass content satisfies a condition analogous to (3.2) (where the parameter a_0 in the expression for φ is replaced by a_{30}), we have two values of the increment for every value of the wave number. If the mass content of the gas is sufficiently large ($g_0<g_*$), the relation becomes single-valued and, as the wave number varies from zero to infinity, the increment increases from some ω_0 to ω_∞ where ω_0 is a root of the equation

$$d_2^2(y_2^2/\beta_2-\Sigma_2)+d_3^2(y_3^2g_0/\beta_3-\Sigma_3+(1-g_0)H_g/\gamma_3)=0$$

Fig.3 shows schematically the dependence of the increment on the wave number. The solid line refers to a liquid containing vapour bubbles $(\alpha_{30}=0)$, the dashed line to $g_0>g_*$ and the dot-dash line to $g_0<g_*$.

Eq.(3.7) can be generalized to the case of a continuous bubble-size distribution. Let us introduce the bubble-size distribution function $f(a_0)$ such, that the volume content of the bubbles $d\alpha$ whose radii vary from a_0 to a_0+da_0, is determined from the relation $d\alpha=(1-\alpha_{10})f(a_0)\,da_0$. Then substituting $\alpha_{i0}=(1-\alpha_{10})f(a_{i0})\Delta a_{i0}$ into (1.8), passing to the limit as $\Delta a_{i0}\to 0$, using the simplification made in deriving (3.7) and assuming that $2\sigma/a_0\ll p_{10}$, $1-g_0\ll 1$, we obtain

$$\left(\frac{Kd}{\omega}\right)^2+\int_{a_{20}}^{a_{30}}f(a_0)\left[\frac{\omega a_0^2}{\beta\varkappa_1^{(T)}}-\frac{1}{\gamma p_{10}}\left(p_{v0}-p_{10}-\frac{4\sigma}{3a_0}\right)\right]^{-1}da_0=0 \qquad (3.8)$$

The above equation for ω has no positive roots when $p_{v0} < p_{10}$, nor when $p_{v0} > p_{10}$ provided that the bubble radii satisfy the condition

$$a_0 < \frac{4}{3} a_\sigma, \quad a_\sigma = \sigma/(p_{v0} - p_{10}) \tag{3.9}$$

Therefore the mixture will be stable under these conditions also. The largest radius of the equilibrium bubbles in the case of a superheated liquid $(p_{v0} > p_{10})$ is found from the relation $a_* = 2a_\sigma$.

Thus the vapour-gas-liquid mixture containing bubbles, underheated with respect to the saturation pressure determined at the flat boundary of separation of the phases, is always stable. The superheated mixture is stable if the bubbles are sufficiently small and satisfy the condition (3.9).

Therefore, the simultaneous action of capillary phenomena and phase transitions may lead to violation of the stability of vapour-liquid mixtures containing bubbles, and the pace of the instability in question will basically be limited by the temperature imbalance in the liquid.

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A GENERAL SOLUTION OF THE STATIC PROBLEM OF THE THEORY OF ASYMMETRICAL ELASTICITY*

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A general solution of the homogeneous static relations of the theory of asymmetric elasticity is constructed. The passage to the solution of the classical (symmetric) theory of elasticity is shown, and the form of the general solution for the plane problem is derived.

Certain modifications to the general solution of the equations of equilibrium in the theory of elasticity serve as a basis for formulating various different expressions for the Castigliano functional in the stress functions /1/.

1. When the mass forces and moments are omitted, the static relations of the theory of asymmetric elasticity have the form /2/

$$\nabla \cdot \mathbf{T} = 0, \quad \nabla \cdot \mathbf{M} - \boldsymbol{\epsilon} \cdot\cdot \mathbf{T} = 0 \tag{1.1}$$

where $\mathbf{T}, \mathbf{M}$ are the asymmetric stress and couple stress tensors, respectively, $\boldsymbol{\epsilon}$ is the Levi-Civita tensor and ∇ is the Hamiltonian operator.

Consider the first relation of (1.1). We know /2/ that a tensor whose divergence is equal to zero can be represented in terms of the curl of another tensor. We therefore write ($\mathbf{P}$ is an arbitrary differentiable second-rank tensor)

$$\mathbf{T} = \nabla \times \mathbf{P} \tag{1.2}$$

Relation (1.2) satisfies the first relation of (1.1) identically. Substituting (1.2) into the second relation of (1.1) and taking into account the validity of the transformation

$$\boldsymbol{\epsilon} \cdot\cdot \nabla \times \mathbf{P} = \nabla \cdot \mathbf{I} \cdot\cdot \mathbf{P} - \mathbf{P} \cdot \nabla$$

we can write the second relation of (1.1) in the form

$$\nabla \cdot (\mathbf{M} + \mathbf{P}^T - \mathbf{I} \cdot\cdot \mathbf{P}) = 0 \tag{1.3}$$

where $\mathbf{I}$ is a unit second-rank tensor and T denotes transposition.

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